The Impact of Default on Tax Shield Valuation

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Abstract In this paper we develop a model to value debt related tax savings and associated yield rates for debt in a setting where future cash flows are uncertain and follow a stochastic diffusion process. By explicitly modeling a default trigger we find that tax shield values in standard DCF valuation formulas are too high as they do not correctly incorporate the risk of default. Furthermore, we are able to endogenously derive risk-adjusted yields.

Keywords: Discounted Cash Flow, Tax Shield, Default, Firm Valuation, Yield Rates
JEL classification: G12, G31, G32, G33

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1 Introduction

Since *Modigliani* and *Miller* (1958), (1963) and *Miles* and *Ezzell* (1980) the valuation of corporate tax shields is one of the most prominent components of the Discounted Cash Flow (DCF) and capital budgeting literature. Event though the tax shield valuation procedure is a broadly used tool in practice and academia it is subject to an ongoing and widespread debate. One recent example is the discussion on the slicing approach to value the tax shield by *Liu* (2009) and *Qi* (2011). A broad literature stream aims at determining tax-adjusted discount rates for the valuation of the tax shield (see for example *Cooper* and *Nyborg* (2008) and *Molnár* and *Nyborg* (2011)). Besides this, there are several articles dealing with misunderstandings about the correct formulas to value the tax shield (see for example the discussion between *Fernandez* (2004), *Fieten et al.* (2005), *Arzac* and *Glosten* (2005) and *Cooper* and *Nyborg* (2006) as well as the more recent discussion between *Massari*, *Roncaglio* and *Zanetti* (2007) and *Dempsey* (2011)).

Despite the aforementioned debate the incorporation of risky debt into the tax shield valuation is not finally resolved. The classic tax shield valuation procedures presume that the risk of default is satisfyingly incorporated by risk-adjusting the respective discount factor of the tax shield and controlling for the performed financing policy (constant debt or leverage). The standard DCF[1] literature suggests that the cost of debt accounts for all possible default risks such as the loss of tax shields or additional bankruptcy costs. Moreover, the tax shield literature focussing on the determination of risk adjusted discount rates for valuing the tax shield usually assumes an exogenously given default without directly modeling it via an explicit default trigger. This leads to a fundamental drawback, which is that the probability of default has to be exogenously assumed. As a consequence the derived risk-adjusted discount rates are not depending on the firm's endogenous default risk.

The central contribution of this paper is that, we extend the tax shield valuation

[1]Notable exceptions are for example *Koziol* (2013) or *Couch*, *Dothan* and *Wu* (2012).

procedure under the assumption of a constant leverage financing policy as proposed by *Miles* and *Ezzell* (1980) to a generalized setting in which default risk and possible bankruptcy costs are explicitly incorporated. By setting a default trigger and relating a potential default to the evolution of the future free cash flows of the firm we provide a tractable approach for determining period-specific risk-adjusted discount rates. Additionally, we show that the risk properties of the tax shield and the debtholder's bond differ. In previous models the value of the tax shield derived from the classical valuation formulas by *Miles* and *Ezzell* (1980) and *Arzac* and *Glosten* (2005) is reported too high. This is the case as the inherent risk of default of debtholders is transferred one-to-one on the tax shield in these models.

The rest of the paper is organized as follows. In section 2 we describe the basic model setup and all relevant assumptions. Furthermore, our approach to model uncertainty through the stochastic diffusion process is described in detail. In section 3 we derive the valuation formula for the risk adjusted discount rates for expected value of debt and the debt related tax savings. Section 4 provides a numerical example of our model as well as a sensitivity analysis. Section 5 concludes the paper.

2 The general model setting

Let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a probability space and $[0,T]$ a time interval, where $T \to \infty$ might be possible. The available information at time t, with $t \in [0,T]$, is denoted by the filtration $\mathcal{F}_t$. The market is assumed to be arbitrage free. Furthermore, there exists an to $\mathbb{P}$ equivalent probability measure $\mathbb{Q}$ called the risk-neutral probability measure. Consider a levered firm whose value in t is denoted by V_t^L. The value of the levered firm is according to the adjusted present value approach (APV) identical to the value of the otherwise identical but unlevered firm V_t^U plus the value of the tax shield V_t^{TS}.[2] The firms operations generate in every period t, which usually corresponds

[2]See for example *Myers* (1974).

to one fiscal year, an uncertain free cash flow stream FCF_t^U. The unlevered cost of equity are denoted by r_u. Debt- and equityholder are faced with the firm's risk of bankruptcy; thus, the debtholders receive on outstanding debt D_t in period t a risk-adjusted (nominal) interest rate $Y_{D,t}$. The risk-free rate r_f and the corporate tax rate τ are certain and constant quantities for all periods.

The firms unlevered free cash flows are assumed to follow the subsequent stochastic process

$$d\text{FCF}^U = \mu \ \text{FCF}^U \ dt + \sigma \ \text{FCF}^U \ dW, \tag{2.1}$$

where μ is the expected rate of return, σ the standard deviation and dW_t a Brownian motion on the probability space $(\Omega, \ \mathcal{F}, \ \mathbb{P})$. Equation (2.1) is the continuous time equivalent of the well known growth assumption in corporate valuation given by

$$E\left[\text{FCF}_{t+1}^U | \mathcal{F}_t\right] = (1+g) \cdot \text{FCF}_t^U, \tag{2.2}$$

where g is the expected growth rate of the free cash flows.

Since the GBM is a stochastic process in continuous time and we only need for the typical DCF framework discrete or periodical observations (e.g. annual, semi-annual,...) of the free cash flows we will discuss at this point the time scale in more detail. As depicted in figure 1 we assume discrete observations, $t, \ t{+}1, \ t{+}2, \ ..., t{+}N$, where each interval is of length Δt, with $\Delta t = \frac{T-t}{N}$. Estimates on the free cash flow values are used for the valuation of the levered firm. This procedure enables us on the one hand to develop the corresponding valuation equations in the typical DCF setting and on the other hand to use the computational benefits of a GBM, e.g. the computational efficiency of a normally distributed random variable.

As outlined in the introduction we will perform a risk-neutral valuation procedure to avoid ex ante assumptions regarding the discount rates for valuing the levered firm. It is well known that a GBM under the risk-neutral probability measure $\mathbb{Q}$ is

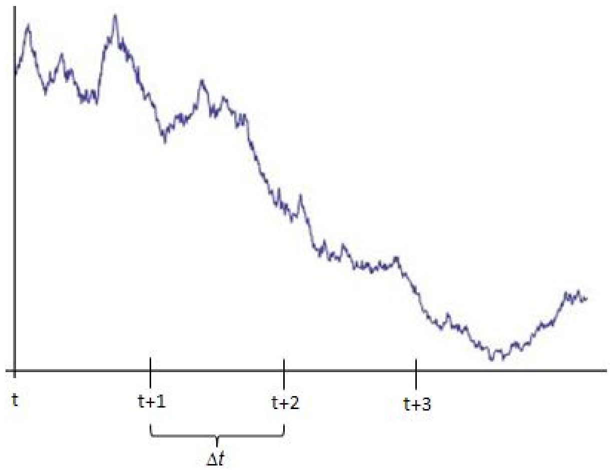


Figure 1: Time scale with a GBM and discrete observations

given by

$$d\text{FCF}^U = r_f \text{ FCF}^U \ dt + \sigma \text{ FCF}^U dW^{\mathbb{Q}}, \tag{2.3}$$

where $dW^{\mathbb{Q}} = dW + \Theta^{\text{GBM}}$ and $\Theta^{\text{GBM}} = \frac{\mu - r_f}{\sigma}$. According to the martingale representation theorem (see *Shreve* (2004)) the discounted FCF^U-process is a martingale under $\mathbb{Q}$. This enables us to price any asset or derivative using the risk-neutral valuation method. Using this procedure the levered firm value according to the APV approach under the risk-neural probability measure $\mathbb{Q}$ can be determined by

$$\begin{aligned} V_t^L &= V_t^U + V_t^{TS} \\ &= \sum_{s=t+1}^{T} \frac{E_{\mathbb{Q}}\left[\text{FCF}_s^U | \mathcal{F}_t\right]}{(1+r_f)^{s-t}} + \sum_{s=t+1}^{T} \frac{E_{\mathbb{Q}}\left[TS_s | \mathcal{F}_t\right]}{(1+r_f)^{s-t}}, \end{aligned} \tag{2.4}$$

where TS_t are the periodic specific debt related tax savings in period t.

This equation is valid for any arbitrary financing policy. In order to explicitly state the (tax shield equation or) periodic specific debt related tax savings it is necessary to assume a specific financing policy. The DCF literature stream differentiates between the following (most commonly used) ones:

1. Autonomous financing (sometimes referred to as passive financing), where D_t is a certain (non-stochastic) quantity for all future periods (see *Modigliani* and *Miller* (1963) or *Myers* (1974)),[3] or

2. Financing based on market values of the levered firm (sometimes referred to as active financing), where the leverage ratio, defined as debt to value ratio ($l = \frac{V_t^L}{D_t}$), is a certain (non-stochastic) quantity for all future periods (see *Miles* and *Ezzell* (1980) and (1985) or *Arzac* and *Glosten* (2005)).[4]

For the remainder of this article we assume the more realistic case of financing based on market values. This financing policy implicitly assumes that the firm raises the amount of debt D_t at time t and raises or redeems the difference between the corresponding debt value D_{t+1} in $t+1$, where both amounts of debt are chosen according to the leverage ratio. Since future amounts of debt are uncertain with regard to the filtration $\mathcal{F}_t$ this implies that (besides the possible risk of bankruptcy) all future periodic specific debt related tax savings are uncertain as well according to $E_{\mathbb{Q}}[TS_s|\mathcal{F}_t] = \tau r_f E_{\mathbb{Q}}[D_s|\mathcal{F}_t]$, with $s > t$.

3 Debt and tax shield valuation in a Miles/Ezzell environment with default

The valuation of a levered firm subject to a possible default is sensitive to the proposed assumptions. Assumptions such as the treatment of a default by the tax authority or assumptions regarding the capital structure after a default must cause changes in the respective valuation formulas. In order to account for these reality-based assumptions we carefully discuss the modeling and valuation implications. For calculating the tax shield value and in turn the risk-adjusted discount rates we first specify in the subsequent sections the payoffs for the debtholders and the tax shield. In this section we start with the assumptions about the firm and continue with those regarding the tax authority.

[3]The constant debt scenario is a special case of this financing policy.
[4]The literature stream usually discusses the special case constant leverage.

Table 1: Payoff of Debtholder and Tax Shield

Payoff	States in $t+1$	
	No Default	Default
Debtholder	Nominal interest + redemption $(1+Y_{D,t})D_t$	Free Cash Flow + Remaining Assets
Tax Shield	Periodic specific tax savings $\tau Y_{D,t}D_t$	-

As in most capital budgeting settings we will consider a limited liability firm. The equity holders are not responsible for the liabilities of the levered firm. The earnings (e.g. EBIT), the investments and the depreciations of a levered firm are identical to the ones of an otherwise identical unlevered firm (e.g. same business risk). This implies, that the performed investment policy of the firm and the depreciation rules of the corresponding accounting authority are independent of the firms level of debt.

As already outlined by *Cooper* and *Nyborg* (2008) it is important to explicitly state assumptions about the tax authorities treatment of a bankrupt firm. In several countries the tax legislation charges taxes on debt relief (acquittance). More commonly, as implicitly assumed by *Miles* and *Ezzell* (1985), the tax authority grants an exception from these tax charges.[5]

To determine the risk equivalent discount factor for the tax shield, we examine the debtholder's payoff and the tax shield as depicted in table 1. The debtholder either receives the contractually specified debt service (no default) or receives the remaining assets of the firm (default). For the tax shield payoff we assume that the firm continues its operations after default all equity financed.

[5] For example the German tax legislative allows for a tax free cancellation of debt when it ensures the continuance of the corporation (compare §227 AO, which was recently confirmed by BFH as of June 14, 2010 – X R 34/08). The US tax authority allows the cancellation of debt income by several reasons which are e.g. for reorganization under chapter 11 or the taxpayer is insolvent (for more details see USC § 108 - income from discharge of indebtedness).

3.1 The value of debt

In this section we explicitly define a default trigger that enables us to determine the expected debt value under consideration of the debtholder's payoff. As a strict consequence we have to distinguish between the contractually fixed interest payments, i.e. the nominal interest payments on debt, and the expected return for the debtholders, i.e. the cost of debt.

A levered firm with an outstanding amount of debt D_t and nominal interest payments of $Y_{D,t} \cdot D_t$ redeems (or raises) in $t+1$ the amount $\Delta D_{t,t+1} = D_t - D_{t+1}$ of debt thus implying an outstanding amount of debt in $t+1$ of D_{t+1}. The levered firm has to settle its debt obligations (i.e. interest payments plus redemption or raise of debt) in $t+1$ by its levered free cash flows FCF^L_{t+1}. Since the following relation between the unlevered and the levered free cash flows is always true[6] $\mathrm{FCF}^L_{t+1} = \mathrm{FCF}^U_{t+1} + \tau \cdot Y_{D,t} \cdot D_t$ the levered firm files for bankruptcy if the following condition holds[7]

$$\mathrm{FCF}^U_{t+1} < (1-\tau)\; Y_{D,t}\; D_t + D_t - D_{t+1} \tag{3.1}$$

or by rearranging

$$\mathrm{FCF}^U_{t+1} < (1-\tau)\; Y_{D,t}\; D_t + \Delta D_{t,t+1}. \tag{3.2}$$

The default criterion implies that the firm defaults as soon as it cannot pay the after tax interests on debt and the debt redemption (if $\Delta D_{t,t+1} > 0$) by its unlevered free cash flows. In particular equation (3.1) reveals that $(1-\tau)\; Y_{D,t}\; D_t + D_t$ has to be paid by the free cash flows of $t+1$ and the new debt outstanding in $t+1$. However, this relationship implicitly assumes that the debtholders provide to the firm an

[6] For the derivation see Appendix A.

[7] This default criterion can be regarded as default due to illiquidity. Illiquidity is one common reason for default in realistic scenarios. In the case of illiquidity debtholders such as banks tend to demand immediate repayment of the overall credit due to a covenant breach.

debt amount of D_{t+1} even in the case of default and thereby ignoring potential reconsiderations on sides of the debtholders.

This property is not a direct implication of the default criterion. As a matter of fact this is a direct consequence of the in the DCF approach typically assumed constant leverage or market value based financing policy. The assumption of this financing policy, that l is a certain quantity in every future period, has a counter-intuitive implication: even in the case of default the debtholders provide to the firm an debt amount of $D_t = l \cdot V_t^L$. For this reason we explicitly assume that after default the firm continues its operations all equity financed. One possible scenario would be that the debtholders take over the firm and continue its operations. As a consequence, the firm pursues a financing policy based on market values until the firm defaults.

This implies in the case of no default, that D_t is thereby just a linear function of the unlevered free cash flows and is in any case given by[8]

$$E_{\mathbb{Q}}\left[D_t\right] = l \cdot \sum_{u=t+1}^{T} \frac{E_{\mathbb{Q}}\left[\mathrm{FCF}_u^U\right]}{\left((1+r_f)\left(1-\frac{\tau \cdot r_f \cdot l}{1+r_f}\right)\right)^{u-t}}. \tag{3.3}$$

At first glance, without forestalling, it might be unreasonable to determine the debt value based upon a "going-concern" value of the levered firm. Nevertheless, a tractable DCF like model for valuing the tax shield, that builds upon the assumption to determine D_t according to the expected "going-concern" value of the levered firm results in elegant valuation formulas.

In order to capture the negative effects of a possible default we introduce bankruptcy costs similar to the standard capital structure models (see for example *Leland* (1994) or *Goldstein*, *Ju* and *Leland* (2001)). In the case of default a fraction $1-\alpha$ of V_t^U, with $0 \leq \alpha \leq 1$ is lost due to indirect bankruptcy costs[9], implying

[8]For the derivation see Appendix B.

[9]As shown by several empirical studies indirect bankruptcy costs account in comparison to the direct bankruptcy costs for a significant amount of the value loss of bankrupt firms. See *Altman* (1984) or for a more recent study *Reimund*, *Schwetzler* and *Zainhofer* (2009).

an unlevered firm value after default of $V_{t+1}^{U,B} = \alpha V_t^U$. This leaves the debtholders in case of no default with the full debt service (interest plus redemption) and in case of default with the free cash flow in $t+1$ and the operating business under consideration of indirect bankruptcy costs. Therefore, the payoff of the debtholder (POD_{t+1}) is given by

$$\mathrm{POD}_{t+1} = \begin{cases} (1+Y_{D,t}) \cdot D_t & \text{, if } \mathrm{FCF}_{t+1}^U \geq (1-\tau)\ Y_{D,t}\ D_t + \Delta D_{t,t+1}, \\ \mathrm{FCF}_{t+1}^U + V_{t+1}^{U,B} & \text{, if } \mathrm{FCF}_{t+1}^U < (1-\tau)\ Y_{D,t}\ D_t + \Delta D_{t,t+1}. \end{cases} \tag{3.4}$$

Equation 3.3 enables us to rewrite (3.2) to

$$\mathrm{FCF}_{t+1}^U < \frac{1}{\gamma_{t+1}}(1-\tau)\ Y_{D,t}\ D_t + D_t, \tag{3.5}$$

$$\text{where } \gamma_{t+1} = 1 + l \cdot \sum_{u=t+2}^{T} \frac{(1+g_{\mathbb{Q}})^{u-(t+1)}}{\left((1+r_f)\left(1-\frac{\tau \cdot r_f \cdot l}{1+r_f}\right)\right)^{u-(t+1)}}. \tag{3.6}$$

Additionally, in the case of default the payoff can be easily rearranged to

$$\mathrm{FCF}_{t+1}^U + V_{t+1}^{U,B} \tag{3.7}$$

$$= \mathrm{FCF}_{t+1}^U + \alpha V_{t+1}^U = \mathrm{FCF}_{t+1}^U \underbrace{\left(1 + \alpha \sum_{u=t+2}^{T} \left(\frac{1+g_{\mathbb{Q}}}{1+r_f}\right)^{u-(t+1)}\right)}_{=M_{t+1}}. \tag{3.8}$$

Thus, the payoff to the debtholders will be the multiple M of the free cash flow in the case of default.[10] Consequently, the default trigger in equation (3.2) can be compared to the strike of a standard European option: The firm defaults in the case where FCF_{t+1}^U is smaller than the strike $K = \frac{1}{\gamma_{t+1}}(1-\tau)\ Y_{D,t}\ D_t + D_t$ and in the remaining case continues its operations. In the case of no default the debtholder receives the full debt service (interests plus redemption) and in the case of default

[10] In Appendix C a condition for α will be derived to show when debtholders receive full recovery after bankruptcy. In the following we consider the possibility of losses for the debtholders due to bankruptcy.

the payoff of the debtholder is given by the unlevered free cash flow FCF^U_{t+1} plus the remaining value of the operating business under consideration of indirect bankruptcy costs $V^{U,B}_{t+1}$.

In the remainder of this subsection, we derive the value of debt in t considering a possible default due to illiquidity. For this purpose we combine equation (3.4) and (3.7) to the present value of the (expected) payoff of the debtholders (3.4), given by

$$E_{\mathbb{Q}}[D_t] = e^{-R_f} \int_{-\infty}^{\infty} \mathrm{POD}_{t+1}\ f_{\mathbb{Q}}(\mathrm{FCF}^U_{t+1})\ d(\mathrm{FCF}^U_{t+1}). \tag{3.9}$$

By noting that all values below zero $(-\infty, 0]$ are irrelevant for our considerations and splitting the integral in two parts we may write

$$\begin{aligned} E_{\mathbb{Q}}[D_t] = e^{-R_f} &\int_{K}^{\infty} (1 + Y_{D,t}) \cdot D_t\ f_{\mathbb{Q}}(\mathrm{FCF}^U_{t+1})\ d\mathrm{FCF}^U_{t+1} \\ + e^{-R_f} &\int_{0}^{K} \mathrm{FCF}^U_{t+1} + V^{U,B}_{t+1}\ f_{\mathbb{Q}}(\mathrm{FCF}^U_{t+1})\ d\mathrm{FCF}^U_{t+1}, \end{aligned} \tag{3.10}$$

where $f_{\mathbb{Q}}(\mathrm{FCF}^U_{t+1})$ denotes the density function of FCF^U_{t+1}. Following Appendix D the solution of equation (3.10) and therefore the value of debt considering a possible default and a financing policy based on market values is given by

$$E_{\mathbb{Q}}[D_t] =(1 + Y_{D,t}) \cdot D_t \cdot e^{-R_f(s-t)} \cdot \mathrm{N}(d_2) + M_{t+1} \cdot \mathrm{FCF}^U_t \cdot \mathrm{N}\left(-d_1\right), \tag{3.11}$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{\mathrm{FCF}_t^U}{K}\right) + \left(R_f + \frac{1}{2}\sigma^2\right)\cdot(s-t)}{\sigma\sqrt{s-t}} \\
d_2 &= d_1 - \sigma\sqrt{s-t} \\
M_{t+1} &= \left(1 + \alpha \sum_{u=t+2}^{T}\left(\frac{1+g_{\mathbb{Q}}}{1+r_f}\right)^{u-(t+1)}\right) \\
K &= \frac{1}{\gamma_{t+1}}(1-\tau)\ Y_{D,t}\ D_t + D_t \\
\gamma_{t+1} &= 1 + l \cdot \sum_{u=t+2}^{T} \frac{(1+g_{\mathbb{Q}})^{u-(t+1)}}{\left((1+r_f)\left(1-\frac{\tau\cdot r_f\cdot l}{1+r_f}\right)\right)^{u-(t+1)}} \\
s &= t+1 \\
R_f &= \ln(1+r_f)\ \text{, is the continuous risk free interest rate.}
\end{aligned}
$$

This valuation formula enables us to derive the (nominal) interest $Y_{D,t}$ that the debtholders charges based upon $E_{\mathbb{Q}}[D_t] = D_t$ in order to get compensated for the default risk.

3.2 The payoff of the tax shield

In this section, we first discuss in more detail the payoff of the tax shield for both the no default and the default case. Second, we derive a valuation equation based upon our findings which enables us to find the correct discount rate. We assume the scenario that after default the firm continues its operations all equity financed. Such a scenario usually occurs by a debt-to-equity swap or through the principles of the bankruptcy proceedings. Taking these preconditions and the default trigger outlined in equation (3.1) into account, the payoff of the tax shield is given by

$$
\mathrm{POTS}_{t+1} = \begin{cases} \tau \cdot Y_{D,t} \cdot D_t & \text{, if } \mathrm{FCF}_{t+1}^U \geq K \\ 0 & \text{, if } \mathrm{FCF}_{t+1}^U < K. \end{cases} \tag{3.12}
$$

With this explicit tax shield payoff in the case of default, we imply that the debtholders either continue operating the defaulted firm or sell the operating assets to a new equity investor. For example after selling the firm to a new equity investor and restructuring, the firm could continue to pursue a financing policy based on market values. Nevertheless, the tax shield value for the debt- and equity holders before restructuring is in the case of default zero. Consequently, the payoff of the tax shield differs from that of the debtholders.

From the payoff of the tax shield given in (3.12) the present value can be calculated via

$$E_{\mathbb{Q}}[TS_t^{PV,t+1}] = e^{-R_f} \int_{-\infty}^{\infty} \text{POTS}_{t+1} f_{\mathbb{Q}}(\text{FCF}_{t+1}^U)\ d(\text{FCF}_{t+1}^U), \tag{3.13}$$

where again all values below zero $(-\infty, 0]$ are irrelevant for our considerations and $f_{\mathbb{Q}}(\text{FCF}_{t+1}^U)$ is again the density function of FCF_{t+1}^U. After separating the integral by

$$\begin{aligned} E_{\mathbb{Q}}[TS_t^{PV,t+1}] =& e^{-R_f} \int_{K}^{\infty} \tau \cdot Y_{D,t} \cdot D_t\ f_{\mathbb{Q}}(\text{FCF}_{t+1}^U)\ d(\text{FCF}_{t+1}^U) \\ &+ e^{-R_f} \int_{0}^{K} \quad 0 \quad f_{\mathbb{Q}}(\text{FCF}_{t+1}^U)\ d(\text{FCF}_{t+1}^U), \end{aligned} \tag{3.14}$$

we can make use of the fact that the second integral has a value of zero. The first integral is over a constant and has been already derived in appendix D. We obtain for the value of the tax shield the following relation

$$E_{\mathbb{Q}}[TS_t^{PV,t+1}] = e^{-R_f}\ \tau \cdot Y_{D,t} \cdot D_t \cdot \text{N}(d_2), \tag{3.15}$$

where d_2 has been already determined in equation (3.11). This equation for valuing the tax shield considering default allows us to calculate the risk-adjusted tax-shield value. The formula accounts for this by using $N(d2)$, the probability that the firm does not default, multiplied with the periodic specific tax deductible interest payments based upon the contractually fixed interest rate $Y_{D,t}$. This interest rate is determined upon our considerations in section 3.1. By comparing equation (3.15) with the standard DCF tax shield formula for a financing policy based on market values ($\frac{\tau Y_{D,t} D_t}{1+Y_{D,t}}$; see *Miles* and *Ezzell* (1980) in combination with *Molnár* and *Nyborg* (2011)), we observe that the tax shield increases with the probability for survival.

4 Implications for the valuation of the tax shield

In this section we determine the impact of a default according to the specified trigger in (3.1) on the value of the tax shield. We calculate the present value of debt $E_{\mathbb{Q}}[D_t]$ considering financing based on market values and the payoff of the debtholders. Finally, we compare the calculated value of the tax shield with its potential values according to the classical tax shield formulas of *Miles* and *Ezzell* (1980) and draw conclusions for the discount rate of the tax shield.

In order to exemplify the impact of default on the value of the tax shield we provide a numerical example throughout this section. We analyze a firm with a limited lifetime of $T = 15$ and an initial free cash flow in t of 100. The growth rate under $\mathbb{Q}$ is equal to the risk-free rate and amounts to 3%, $g_{\mathbb{Q}} = r_f = 0.03$. The firm targets a constant leverage ratio of $l = 0.25$, which is an average estimate for firms operating in the $G7$ states. Additionally, we assume a standard deviation of the free cash flows of $\sigma = 0.15$, depicting thereby a moderate fluctuation of the firm's free cash flows. The corporate tax rate is assumed to be constant and amounts to $\tau = 35\%$.

We start our analysis by calculating the value of debt according to (3.3). For the aforementioned parameters $E_{\mathbb{Q}}[D_t]$ amounts to 382.76. It is now possible to compare the results from (3.3) to that of (3.11), where both equations must strictly yield the same result. Equation (3.11) incorporates the potential risk of default and the associated losses. Therefore, the debtholders will be compensated for the associated default risk by an appropriate contractually fixed interest rate $Y_{D,t}$.

From a technical perspective we notice that we cannot easily rearrange equation (3.11) for $Y_{D,t}$ due to the cumulative normal distributions. Nevertheless, a numerical solution can be easily calculated. Table 2 shows for various $Y_{D,t}$ ranging from 4.5% to 8% the implied expected values of debt while the other parameters of the initial example are kept constant. With increasing $Y_{D,t}$ we observe increasing levels of debt. This important result documents that the default risk due to higher levels of debt is appropriately compensated by higher contractually fixed interest rates. For the parameters of our initial example the debtholders would set $Y_{D,t} = 7.2605\%$.[11]

Table 2: Expected value of debt for given $Y_{D,t}$

$Y_{D,t}$	K	$N(d_2)$	$N(-d_1)$	$E_{\mathbb{Q}}[D_t]$
8.00 %	88.15	0.8322	0.13	384.48
7.50 %	87.88	0.8373	0.13	383.32
7.00 %	87.61	0.8423	0.12	382.14
6.50 %	87.33	0.8473	0.12	380.93
6.00 %	87.06	0.8521	0.12	379.71
5.50 %	86.79	0.8569	0.11	378.47
5.00 %	86.52	0.8616	0.11	377.21
4.50 %	86.25	0.8662	0.10	375.92

The tax shield value subject to a possible default and loss of future tax shields given in equation (3.15) amounts to

$$E_{\mathbb{Q}}[TS_t^{PV,t+1}] = e^{-0.0296} \cdot 0.35 \cdot 0.072605 \cdot 382.76 \cdot 0.83978 = 7.93. \tag{4.1}$$

For the parameters of the initial example the standard DCF value of the tax shield $V^{TS,ME}$ without the explicit modelling of default amounts to

$$V_t^{TS,ME} = \frac{0.35 \cdot 0.072605 \cdot 382.76}{1 + 0.072605} = 9.068. \tag{4.2}$$

By comparing the calculated tax shield values $E_{\mathbb{Q}}[TS_t^{PV,t+1}]$ and $V_t^{TS,ME}$ we observe

[11]This solution can be easily obtained via bisection.

that the standard procedure for valuing tax shields underestimates the consequences of a possible default.[12]

In the next step, we calculate the respective discount rate for the tax shield via the following return equation

$$r_{TS} = \frac{\tau Y_{D,t} D_t}{E_{\mathbb{Q}}[TS_t^{PV,t+1}]} - 1. \tag{4.3}$$

Thereby, we assume that the discount rate is defined as a conditional expected return. In the example, the resulting discount rate for the tax shield subject to default is calculated by

$$r_{TS} = \frac{0.35 \cdot 0.072605 \cdot 382.76}{7.93} - 1 = 0.2266. \tag{4.4}$$

In the remainder of this section, we outline the sensitivity of the pricing algorithm by varying the parameters leverage (l), the standard deviation of the free cash flows (σ) and the recovery rate (α). Thereby, we analyze every parameter with respect to its impact on the level of debt and the contractually fixed interest rate $Y_{D,t}$.

We start with the impact of the leverage. Figure 2 depicts the calculated debt values and the corresponding promised yields for various leverage ratios. Since the assumption of a certain (or constant) leverage ratio implies that the value of debt is a linear derivative of the levered firm value, it is obvious that the total amount of debt increases with leverage. With increasing total amounts of debt the implied probability to survive $N(d_2)$ becomes smaller and the debtholders demand a higher promised yield for compensation of the increased default risk.

In the next step of the sensitivity analysis we consider the impact of the volatility of the free cash flows (see Figure 3). Due to the construction of the financing policy on market values, $D_t = l \cdot V_t^L$ the volatility has no direct impact on the value of the levered firm and therefore the value of debt is independent of the volatility as well.

[12]Notice that the application of the tax shield formula according to *Sick* (1990) (taxes on debt relief apply) yields a value of $\frac{\tau r_f D_t}{1+r_f} = 3.9$.

Figure 2: Resulting $Y_{D,t}$ for different choices of leverage l.

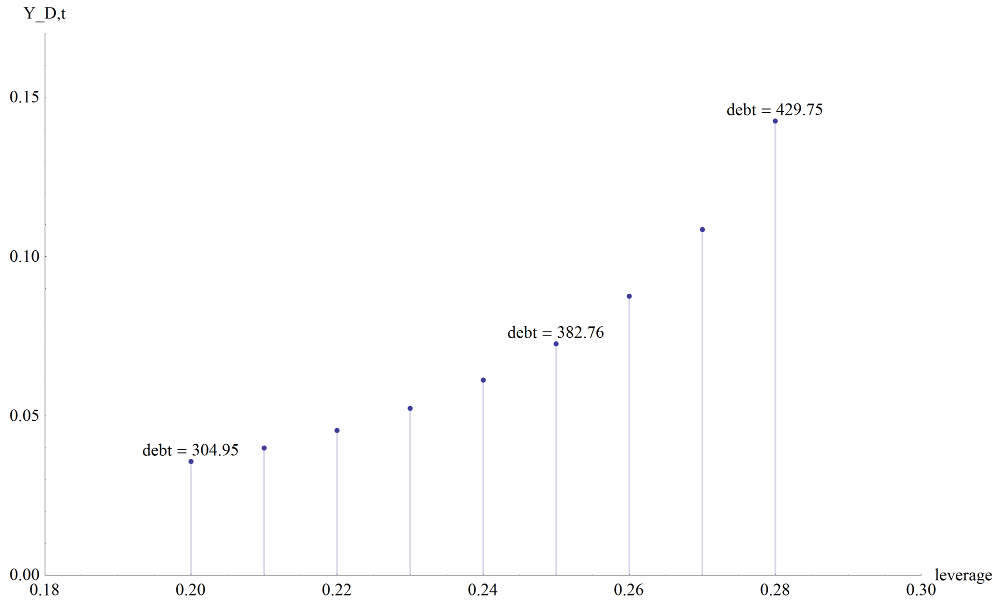
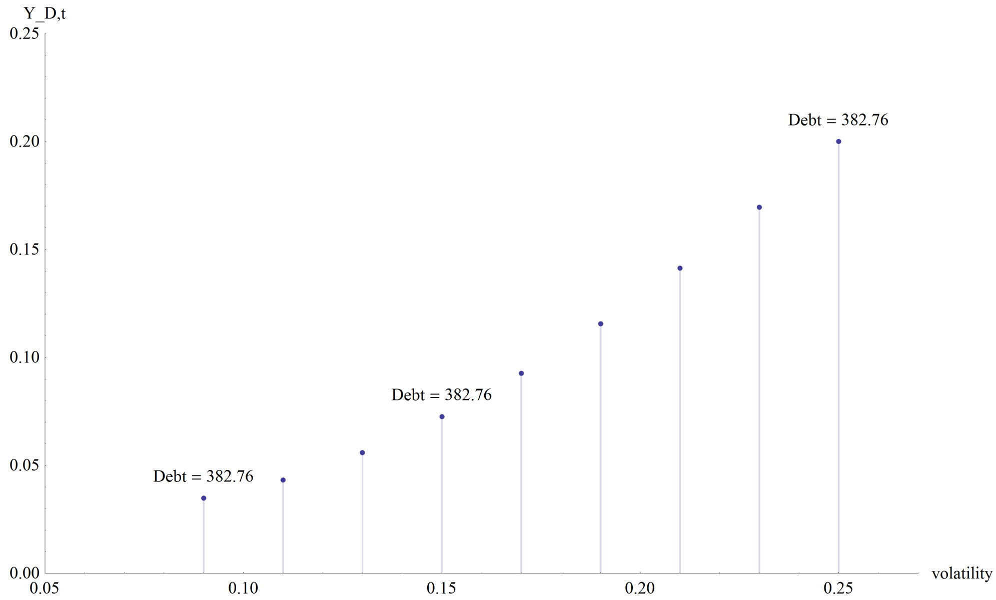
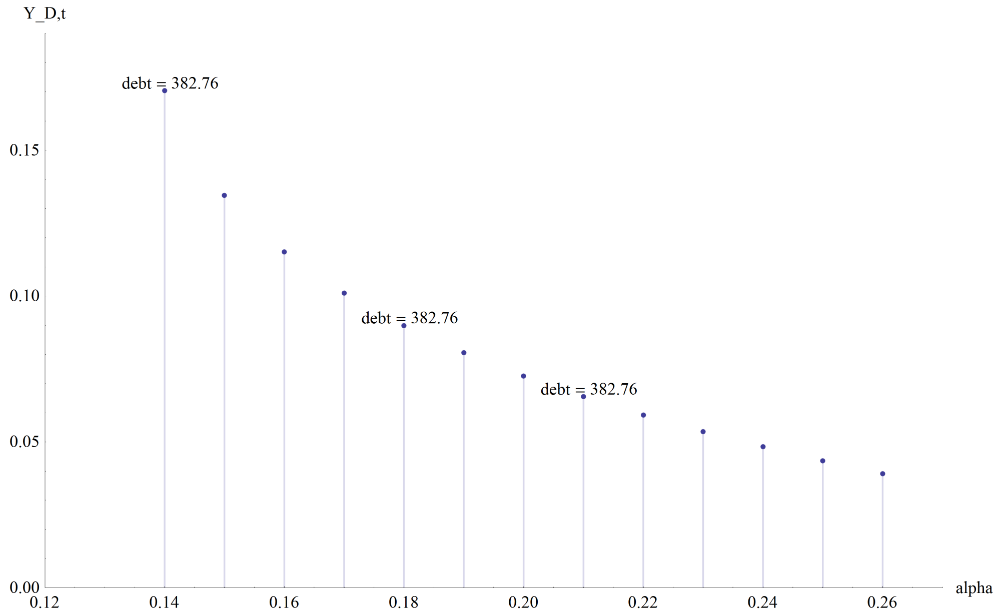


Figure 3: Resulting $Y_{D,t}$ for different choices of volatility σ.



Even though the value of debt remains unaffected, the contractually fixed interest rate $Y_{D,t}$ increases with respect to the standard deviation. Clearly, this result is intuitive, a high volatility implies that the probability to survive decreases.

Figure 4: Resulting $Y_{D,t}$ for different choices of the recovery rate α.



In the last step we discuss the impact of the recovery rate α upon the probability to survive $N(d_2)$ and the promised yield $Y_{D,t}$. As mentioned before the value of debt is independent of α. As depicted in Figure 4 a higher recovery rate leads to a lower promised yield $Y_{D,t}$. Clearly, in the case of default the expected payoff for the debtholders increases with respect to α and therefore the risk compensation is smaller. The recovery rate has an upper boundary (see equation (C.5)), where the debtholders suffer no loss in the case of the default. For the parameters of this example the upper boundary α_{max} amounts to 26%.

5 Conclusion

In this article we derive a general model for tax shield valuation considering the possibility of a default under financing based on market values. The general formula models the possibility of a default by explicitly using the default trigger illiquidity. By doing so, we endogenously determine the possibility of default without imposing any assumption with respect to an exogenous given probability of default. Based upon the probability of default, or inversely the probability to survive, the contractually fixed interest on debt for compensating the debtholders is computed. This is important, because the promised yield should fit to the firm's implied probability of default for practical valuation purposes. Additionally, we have shown how to calculate the risk-adjusted discount rate for the valuation of tax shields.

The presented tax shield valuation formula is equal to the standard DCF tax shield formula for the case of no possibility of default or a firm free of the risk of default. In this case the promised yield corresponds to the risk free rate.

However, for the case of a risky firm, we have illustrated in a simple DCF setting how to endogenously determine the probability of default, the implied interest on debt and the tax shield value subject to default.

A Derivation of FCF_t^L

Using the assumptions X (acquittance / remission of debt) and Y (equity financing ex-post bankruptcy), we derive the relations for TAX^L and FCF^L. Since we assume that the tax authority does not impose a tax on a possible acquittance the levered firm has to pay taxes equivalent to

$$\mathrm{TAX}_t^L = \tau \cdot (\mathrm{EBITDA}_t - \mathrm{Depr}_t - r_D \cdot D_{t-1})\,. \tag{A.1}$$

By noting that the taxes paid by unlevered firm TAX^U only differ by the interest payments $r_D \cdot D_{t-1}$ the levered free cash flows can be calculated by

$$\begin{aligned}
\mathrm{FCF}_t^L &= \mathrm{EBITDA}_t - \mathrm{INV}_t - \mathrm{TAX}_t^L \\
&= \mathrm{EBITDA}_t - \mathrm{INV}_t - \tau \cdot (\mathrm{EBITDA}_t - \mathrm{Depr}_t - r_D \cdot D_{t-1}) \\
&= \mathrm{EBITDA}_t - \mathrm{INV}_t - \mathrm{TAX}_t^U + \tau \cdot r_D \cdot D_{t-1} \\
&= \mathrm{FCF}_t^U + \tau \cdot r_D \cdot D_{t-1}.
\end{aligned} \tag{A.2}$$

B Derivation of V_t^L under $\mathbb{Q}$

In order to determine an expression for the WACC under the risk-neutral probability measure $\mathbb{Q}$ we start with the value of a levered firm under $\mathbb{Q}$:

$$V_t^L = \frac{E_{\mathbb{Q}}\left[V_{t+1}^L + \mathrm{FCF}_{t+1}^L | \mathcal{F}_t\right]}{1 + r_f}.$$

Substituting $\mathrm{FCF}_{t+1}^L = \mathrm{FCF}_{t+1}^U + \tau \cdot r_f \cdot D_t$ and $D_t = l \cdot V_t^L$ and rearranging yields:

$$V_t^L = \frac{E_{\mathbb{Q}}\left[V_{t+1}^L + \mathrm{FCF}_{t+1}^U | \mathcal{F}_t\right]}{1 + r_f} + \frac{\tau \cdot r_f \cdot l \cdot V_t^L}{1 + r_f} \tag{B.1}$$

$$= \frac{E_{\mathbb{Q}}\left[V_{t+1}^L + \mathrm{FCF}_{t+1}^U | \mathcal{F}_t\right]}{(1 + r_f)\left(1 - \frac{\tau \cdot r_f \cdot l}{1 + r_f}\right)} \tag{B.2}$$

By repeating this procedure until time T we are able to determine the following expression for the levered firm value

$$V_t^L = \sum_{s=t+1}^{T} \frac{E_{\mathbb{Q}}\left[\mathrm{FCF}_s^U\right]}{\left((1+r_f)\left(1-\frac{\tau\cdot r_f\cdot l}{1+r_f}\right)\right)^{s-t}}. \tag{B.3}$$

C The limit of debtholder's payoff in bankruptcy state

According to equation (3.7) it may be possible that the debtholders receive full recovery in the states of bankruptcy. The payoff to debtholders can be equal (or larger) to the outstanding liabilities in equation (3.4). The necessary condition is $\mathrm{FCF}_{t+1}^U + V_{t+1}^{U,B} \geq (1+Y_{D,t})\cdot D_t$.

The constraint can be solved for α:

$$\mathrm{FCF}_{t+1}^U + V_{t+1}^{U,B} \geq (1+Y_{D,t})\cdot D_t \tag{C.1}$$

$$\mathrm{FCF}_{t+1}^U \cdot \left(1+\alpha\sum_{u=t+2}^{T}\left(\frac{1+g_{\mathbb{Q}}}{1+r_f}\right)^{u-(t+1)}\right) \geq (1+Y_{D,t})\cdot D_t \tag{C.2}$$

$$\mathrm{FCF}_{t+1}^U \geq \frac{(1+Y_{D,t})\cdot D_t}{\left(1+\alpha\sum_{u=t+2}^{T}\left(\frac{1+g_{\mathbb{Q}}}{1+r_f}\right)^{u-(t+1)}\right)}. \tag{C.3}$$

Now substitute the strike $\mathrm{FCF}_{t+1}^U = \frac{1}{\gamma_{t+1}}(1-\tau)\ Y_{D,t}\ D_t + D_t$

$$\frac{1}{\gamma_{t+1}}(1-\tau)\ Y_{D,t}\ D_t + D_t \geq \frac{(1+Y_{D,t})\cdot D_t}{\left(1+\alpha\sum_{u=t+2}^{T}\left(\frac{1+g_{\mathbb{Q}}}{1+r_f}\right)^{u-(t+1)}\right)} \tag{C.4}$$

and solve for α

$$\alpha \geq \frac{\frac{(1+Y_{D,t})\cdot D_t\cdot\gamma_{t+1}}{(1-\tau)\ Y_{D,t}\ D_t+D_t} - 1}{\sum_{u=t+2}^{T}\left(\frac{1+g_{\mathbb{Q}}}{1+r_f}\right)^{u-(t+1)}} \tag{C.5}$$

If the condition holds the debtholders will charge the risk free interest rate $Y_{D,t} = r_f$ as they will receive full recovery in bankruptcy and non-bunkruptcy states.

D Derivation of the integrals

For solving the two integrals given in equation (3.10) it is important to note

1. In the first term $(1 + Y_{D,t}) \cdot D_t$ is in t a constant and therefore can be factored out.

2. The second integral can be solved by substituting (3.7) in (3.10)

$$\text{FCF}^U_{t+1} + V^{U,B}_{t+1} = \text{FCF}^U_{t+1} \cdot M_{t+1}. \tag{D.1}$$

Given these simplifications equation (3.10) can be restated for an arbitrary time span $s - t$, with $s > t$, by

$$\begin{aligned} E_{\mathbb{Q}}[D_t] =& (1 + Y_{D,t}) \cdot D_t e^{-R_f(s-t)} \int_K^\infty 1 f_{\mathbb{Q}}(X)\ d(X) \\ &+ M_{t+1} \int_0^K X_s\ f_{\mathbb{Q}}(X)\ d(X). \end{aligned} \tag{D.2}$$

By acknowledging that X_s is lognormally distributed and can be transformed to a standardnormal distributed variable Z, by

$$Z = \frac{\ln X_s - \left(\ln X_t + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)\right)}{\sigma\sqrt{s-t}} \sim \text{N}(0,1), \tag{D.3}$$

the first term in equation (3.10) can be rearranged using the density of the normal distribution $\varphi(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}$ to

$$(1 + Y_{D,t}) \cdot D_t e^{-R_f(s-t)} \int_{\frac{\ln K - \left(\ln X_t + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)\right)}{\sigma\sqrt{s-t}}}^\infty 1\ \varphi(Z)\ d(Z) \tag{D.4}$$

$$= (1 + Y_{D,t}) \cdot D_t e^{-R_f(s-t)} \text{N}\left(\frac{\ln\left(\frac{X_t}{K}\right) + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}\right) \tag{D.5}$$

$$= (1 + Y_{D,t}) \cdot D_t e^{-R_f(s-t)} \text{N}(d_2) \tag{D.6}$$

The solution of the second term is given by

$$M_{t+1} \cdot e^{-R_f(s-t)} \int_0^K X_t \ f_{\mathbb{Q}}(X) \ d(X) \tag{D.7}$$

$$= M_{t+1} \cdot e^{-R_f(s-t)} \int_{-\infty}^{\frac{\ln K - \left(\ln X_t + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)\right)}{\sigma\sqrt{s-t}}} e^{Z\cdot\sigma\sqrt{s-t} + (\ln X_t + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t))} \ \varphi(Z) \ d(Z) \tag{D.8}$$

$$= M_{t+1} \cdot e^{(\ln X_t + \left(-\frac{1}{2}\sigma^2\right)(s-t))} \int_{-\infty}^{\frac{\ln K - \left(\ln X_t + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)\right)}{\sigma\sqrt{s-t}}} e^{Z\cdot\sigma\sqrt{s-t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} \ d(Z) \tag{D.9}$$

$$= M_{t+1} \cdot e^{(\ln X_t)} \int_{-\infty}^{\frac{\ln K - \left(\ln X_t + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)\right)}{\sigma\sqrt{s-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z-\sigma\sqrt{s-t})^2}{2}} \ d(Z) \tag{D.10}$$

$$= M_{t+1} \cdot X_t \int_{-\infty}^{\frac{\ln K - \left(\ln X_t + \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)\right)}{\sigma\sqrt{s-t}} - \sigma\sqrt{s-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z)^2}{2}} \ d(Z) \tag{D.11}$$

$$= M_{t+1} \cdot X_t \ \mathrm{N}\left(\frac{\ln\left(\frac{K}{X_t}\right) - \left(R_f - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}} - \sigma\sqrt{s-t}\right) \tag{D.12}$$

$$= M_{t+1} \cdot X_t \ \mathrm{N}\left(-\frac{\ln\left(\frac{X_t}{K}\right) + \left(R_f + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}\right) \tag{D.13}$$

$$= M_{t+1} \cdot X_t \ \mathrm{N}\left(-d_1\right) \tag{D.14}$$

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